17.1 Background
17.2 f_e, f_o
17.3 F S.

Ch. 17. F S (Fourier series), F I (integral), F T (transform)

17.1. Background & Motivation

I. Background

Expansion by infinite series !!

$$TS\,f = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

polynomial

$$FS = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

trigonometric.

II. Motivation

$$\alpha^2 \nabla^2 u = u_t \rightarrow \sum a \sin \alpha x \cdot e^{-t}$$

applications: diffraction, spectrum/signal analysis, quantum mechanics, data compression, MRI, CT, PET ...

17.2 Odd fxns, Even fxns

$f_o(x)$ $\quad$ $f_e(x)$

I Definition

1. $f(x) = f(-x)$; e.g. x^2

$\rightarrow f_e(x)$

2. $f(x) = -f(-x)$; $f(-x) = -f(x)$

$f_o(x) \rightarrow x^1, x^3, \cdot$

II. operation & properties

1.

① $f_e + f_e \to f_e$ ② $f_o + f_o \to f_o$

③ $f_e + f_o \to ?$, ④ $f_o \times f_o \to f_e$

⑤ $f_e \times f_e \to f_e$ ⑥ $f_o \times f_e \to f_o$

$$\int_{-A}^{A} f_e(x)\,dx = 2\int_{0}^{A} f_e(x)\,dx$$

$$\int_{-A}^{A} f_o(x)\,dx = 0$$

Pf

$$\int_{-A}^{A} f(x)\,dx = \int_{-A}^{0} f(x)\,dx + \int_{0}^{A} f(x)\,dx$$

for example, $f(x) = f_e(x)$; set $x = -t$

x	$-A$	0
t	A	0

; $dx = -dt$

$$\sim\sim = \int_A^0 \underbrace{f(-t)}_{}\,\overset{-f(t)}{}(-dt) = \int_0^A f(t)\,dt - \int_0^A f(x)\,dx$$

$(*)$ becomes $2\int_0^A f(x)\,dx$ #

III.

$$f(x) \overset{\text{decompose}}{=\!=\!=} f_e(x) + f_o(x)$$

Pf

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$= f_e(x) + f_o(x)$$

$$F(x) = \frac{f(x)+f(-x)}{2} \quad ; \quad F(-x) = \frac{f(-x)+f(x)}{2}$$

$$= F(x) = f_e(x)$$

Likewise,

$$G(x) = \frac{f(x)-f(-x)}{2} \quad ; \quad G(-x) = \frac{f(-x)-f(x)}{2}$$

$$= -G(x) = f_o(x)$$

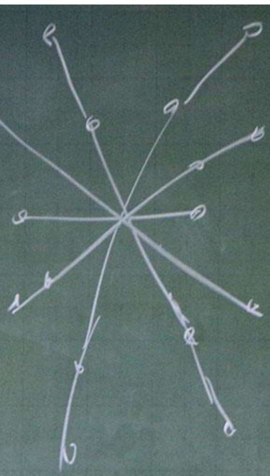
IV. periodic fxns. f_T

$f(x+\underbrace{T}_{\text{period}}) = f(x) \rightarrow f(x)$ is a periodic fxn.

T = period, min. $T \rightarrow$ fundamental

period ; period — temporal domain

*spatial domain



lattice.

17.3 FS of a periodic fxn.

I. FS

For a periodic fxn $f(x)$, with a period 2ℓ, then,

$$FS\, f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \qquad (1)$$

, where

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx \qquad \text{———} \; 2(a)$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx \qquad \text{———} \; 2(b)$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx \qquad \text{———} \; 2(c)$$

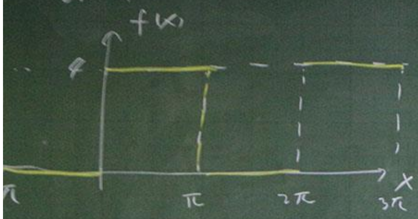
Comments

1. superposition of harmonic wave!!

2. $\sin \omega t$, $\sin(\omega t + \alpha)$ — $\left[\begin{array}{l} \sin \omega t \\ \cos \omega t \end{array}\right.$

$$\sin(\omega t + \alpha) = \underline{\sin \omega t}\cos\alpha + \sin\alpha\,\underline{\cos \omega t}$$

<Ex>



$$P = 2\pi = 2l \rightarrow l = \pi$$

find FS $f(x)$?

Sol

$$FS\, f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \quad \text{———(1)}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} dx = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{Known}} \cos nx\, dx$$

$$= \frac{4}{\pi}\int_0^{\pi} \cos nx\, dx = \frac{4}{\pi}\cdot\frac{1}{n}\sin nx\Big|_0^{\pi}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{4}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{4}{\pi}\cdot\frac{-1}{n}\cos nx\Big|_0^{\pi}$$

$$= \frac{4}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0, & n = \text{even} \\ \frac{8}{n\pi}, & n = \text{odd} \end{cases}$$

$$\therefore FS\, f(x) = 2 + \sum_{n=1,3,5}^{\infty} \frac{8}{n\pi} \sin nx \quad \#$$

$S_1(x) = a_0 = 2$ = ave. value!!

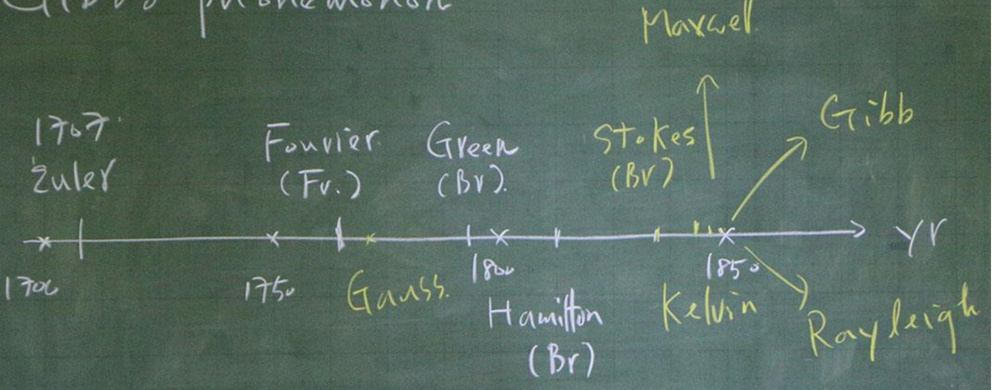
$S_2(x) = a_0 + \frac{8}{\pi}\sin x$

$S_3(x) = a_0 + \frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x$

I. Fourier Convergence Theorem

II. Gibb's phonemonon

PS



IV Euler's formulas (×3)

$$\int_{-l}^{l} \cos\frac{m\pi x}{l} \cos\frac{n\pi x}{l}\, dx = \begin{cases} 0, & m \neq n \\ l, & m = n \end{cases}$$

$$\int_{-l}^{l} \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l}\, dx = \begin{cases} 0, & m \neq n \\ l, & m = n \end{cases}$$

$$\int_{-l}^{l} \cos\frac{m\pi x}{l} \sin\frac{n\pi x}{l}\, dx = 0$$

If

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \quad \text{—— } (1,2)$$

$$\longrightarrow \quad \cos A \cos B = \frac{\cos(A-B) + \cos(A+B)}{2} \quad \text{—— } (3)$$

$$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2} \quad \text{—— } (4)$$

$$\int_{-l}^{l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-l}^{l}\left[\cos\frac{(m+n)\pi x}{l} + \cos\frac{(m-n)\pi x}{l}\right]dx$$

$$= \frac{1}{2}\left[\frac{l}{(m+n)\pi}\sin\frac{(m+n)\pi x}{l} + \frac{l}{(m-n)\pi}\sin\frac{(m-n)\pi x}{l}\right]\Bigg|_{-l}^{l} \quad \text{———} \quad (5)$$

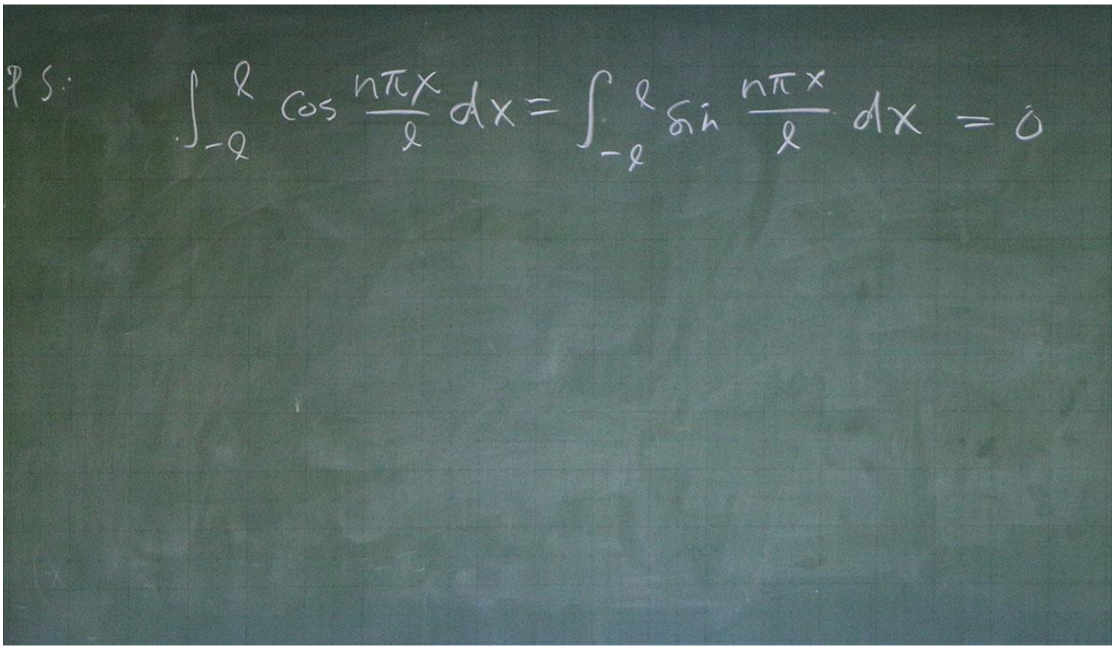
1. $m \neq n$, $(5) = 0$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

2. $m = n$

$$\int_{-l}^{l} \cos^2\frac{n\pi x}{l}\,dx = \int_{-l}^{l}\left(\frac{1+\cos\frac{2n\pi x}{l}}{2}\right)dx$$

$$= \frac{1}{2}\left[x + \underbrace{\sim}_{0}\right]\Bigg|_{-l}^{l} = l.$$


PS:
$\int_{-l}^{l} \cos \frac{n\pi x}{l} dx = \int_{-l}^{l} \sin \frac{n\pi x}{l} dx = 0$


PS

P.S.: $\int_{-l}^{l} \cos\frac{n\pi x}{l}\,dx = \int_{-l}^{l} \sin\frac{n\pi x}{l}\,dx = 0$

P.S. $\int_{a \to -l}^{b \to l} f(x)g(x)\,dx = \begin{cases} 0, & m \neq n \\ \neq 0, & m = n \end{cases}$

$$a_i \cdot a_j = \begin{cases} 0, & i \neq j \\ \neq 0, & i = j \end{cases}$$